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Products and compositions with the Dirac delta function

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Abstract. The need to define pointwise products and compositions with distributions is pointed out in the context of the problems of renormalisation, junction conditions and curved shock waves. Earlier definitions are briefly reviewed, and new definitions are proposed using non-standard analysis. Basic properties are established, and some products and compositions with the delta distribution are explicitly evaluated. With these definitions, the domain of validity of the nonlinear differential equations of classical field theory can be extended to include discontinuous fields, without introducing new phenomenology. As an example, the Rankine–Hugoniot equations are derived from the Euler equations. An immediate application to quantum field theory is pointed out.

1. Introduction

The notion of the Dirac delta function has been rigorously formulated in Schwartz's (1951) theory of distributions, the equivalent theory of generalised functions (Gel'fand and Shilov 1964), and Mikusinski's (1959) theory of operators. In all these theories, pointwise products (i.e. products of two generalised functions of the same 'argument') and compositions (of a distribution, or an operator, with an ordinary function) are irregular operations in the sense of Mikusinski (1961). However, in many concrete situations in physics, such irregular operations arise, and are dealt with, without due regard to rigour. Although this problem has been known for nearly three decades, it remains incompletely solved.

The problem of defining pointwise products of distributions is closely connected with the problem of renormalisation in quantum field theory. The propagators (Green functions) of quantum field theory are well known distributions, and products of these propagators enter into the perturbation expansion of the S matrix. Going over to the momentum representation, the (formal) Fourier transformations of these products lead to divergent integrals. Thus, the so-called divergences of the S matrix can be traced to difficulties in defining the product of two distributions (Guttinger 1955, Bogoliubov and Shirkov 1959).

Earlier attempts to define, or use, pointwise products of distributions (Konig 1954, Guttinger 1955, Gonzalez-Dominguez and Scarfiello 1956, Mikusinski 1961, 1966, Fisher 1971, Thurber and Katz 1974) have, therefore, been inspired by possible applications to the problem of renormalisation. In particular, most of the effort seems to have gone into proving the formula

$$\delta \cdot x^{-1} = -\tfrac{1}{2}\delta' \tag{1.1}$$

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which is of some use in quantum field theory. Similarly, compositions have been defined in only a few simple cases (Lojasiewicz 1957, Fisher 1974, Tewari 1977), and many interesting expressions lie outside the scope of these definitions.

Naturally, these operations, if suitably defined, have a much wider range of applicability. The two examples given below illustrate the general situation that might arise in classical field theory.

1.1. Junction conditions

In general relativity, the exact degree of smoothness that can be assigned to the components of the metric tensor, $g_{\mu\nu}$, is not known. In certain situations it may be physically permissible to choose the $g_{\mu\nu} \in C^0$ (for example, Lanczos 1924, Papapetrou and Hamoui 1968, 1979, Evans 1977). Mathematically, however, this leads to difficulties in view of the usual formulae

$$\begin{aligned} \Gamma_{\mu\nu\sigma} &= \tfrac{1}{2}(g_{\mu\nu,\sigma} + g_{\mu\sigma,\nu} - g_{\nu\sigma,\mu}) \\ R_{\mu\nu} &= \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} \\ R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R &= -\kappa T^{\mu\nu} \\ T^{\mu\nu}{}_{:\nu} &= 0. \end{aligned} \tag{1.2}$$

In particular, if the $g_{\mu\nu}$ were chosen to be discontinuous, as suggested by Raju (1979), the components of the Ricci tensor would involve functions of the form δ^2. Thus, either one has to solve the problem mentioned in the first paragraph, or abandon formulae (1.2) and develop altogether new techniques, that may or may not be reliable.

1.2. Curved shocks

Shock waves arising in practice are usually curved, and the equations of continuity and momentum

$$\begin{aligned} \partial\rho/\partial t + \operatorname{div}(\rho \boldsymbol{v}) &= 0 \\ \rho(\partial \boldsymbol{v}/\partial t) + \rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} &= -\nabla P - \rho\nabla F + \operatorname{div}\mathscr{S}_{\mathrm{v}} \end{aligned} \tag{1.3}$$

(where $\mathscr{S}_{\mathrm{v}}$ is the viscous stress tensor) immediately lead to the above problem if the density ρ, pressure P and velocity $\boldsymbol{v}$ are chosen to be discontinuous at an arbitrary hypersurface.

To solve the above problems, it is necessary to assign some meaning to entities such as $\delta(x)\cdot\delta(x)$, $\delta(f(x))$, etc. Mathematically there may be many ways to do this, but the physical aspect of the problem imposes certain constraints. An arbitrary definition would not be of much value to the physicist, because our continued belief in the equations (1.2) and (1.3), for example, if at all justified, would require fresh phenomenology. A similar difficulty would seem to arise with regard to the formalism of quantum field theory. In the sequel, certain elementary techniques from non-standard analysis are used to obtain a solution to the problem of defining irregular operations in a physically reasonable manner.

Non-standard analysis rigorously incorporates infinitesimals and infinities, and one may add and multiply these exactly like ordinary numbers. There is no danger of reaching absurd conclusions, provided the *final* result is always a standard one, i.e. does not involve infinities or infinitesimals. This is because it has been proved (Robinson

1966, Stroyan and Luxemburg 1976) that *any standard result derived using non-standard techniques is necessarily a valid result,* and, in principle, could have been derived without using these techniques. (In practice, of course, it may be very difficult to obtain a standard proof, and even so the standard proof is usually much lengthier.) Non-standard analysis, therefore, provides an ideal tool because, in both classical and quantum field theory, irregular operations with distributions are required only at an intermediate stage, and the final result must be free from infinities if it is to be physically meaningful.

2. Products

2.1. *Earlier definitions*

We let D, D' denote the space of test functions and distributions respectively. It is well known that the product of $T \in D'$ and $f \in D$ can be defined by

$$\langle Tf, g\rangle = \langle T, fg\rangle \tag{2.1}$$

for any $g \in D$, $\langle T, h\rangle$ denoting the value of the functional T at h. This product is well defined, and the Leibniz formula holds (see, for example, Rudin 1974). Konig (1954) has constructed product spaces of distributions, and mapped these spaces back into D' in a manner that preserves the formula (2.1) in the form

$$\langle TS, g\rangle = \langle T, Sg\rangle. \tag{2.2}$$

It is asserted that (2.2) makes sense on the null space of S, with $TS = 0$ there. Thus, we have

$$\delta(x-a)\delta(x-b) = c_1\delta(x-b) \qquad \delta\delta' = c_2\delta' + c_3\delta \tag{2.3}$$

where c_1, c_2, c_3 are arbitrary constants. The product, in general, is neither commutative nor associative. In fact, there is only one possibility concerning the association of factors in a product of $T_1, T_2, \ldots, T_n$, either $T_1(T_2(\ldots T_n))\ldots)$ or $(\ldots((T_1T_2)\ldots)T_n$. The usual example for the failure of the associative law is

$$x^{-1}(x\delta) = 0 \neq \delta = (x^{-1}x)\delta. \tag{2.4}$$

The main problem with the product, so defined, lies in arbitrariness in the choice of the constants c_1, c_2, c_3. In practice, the choice of the constants is tailored to meet the needs of a particular application. Needless to say, the tailoring does not always fit the physical requirements of the problem, rendering our continued belief in equations of the type (1.2) and (1.3) invalid.

Mikusinski (1961), on the other hand, has proposed a general theory of irregular operations for distributions. If R is an operation defined for test functions, R can be extended to distributions by defining

$$R(f, g, \ldots) = \lim_{n\to\infty} R(\varphi_n, \eta_n, \ldots) \tag{2.5}$$

(where $\{\varphi_n\}, \{\eta_n\}, \ldots$ are fundamental sequences converging to $f, g, \ldots$), provided the sequence $\{R(\varphi_n, \eta_n, \ldots)\}$ is fundamental. The sequence φ_n can be obtained as $f\otimes\delta_n$, where δ_n is a sequence converging to δ, and $\otimes$ denotes convolution. It is asserted that the extension of the operation R, so defined, exists and is unique. Mikusinski (1966) uses this definition to obtain the formula

$$\delta^2 - \pi^{-2}(x^{-1})^2 = -\pi^{-2}x^{-2} \tag{2.6}$$

and also the equality (1.1). The left-hand side of (2.6) is considered as the distributional limit of $\delta_n^2 - \pi^{-2}(x^{-1}\otimes\delta_n)^2$, no meaning being assigned to the individual terms.

One problem with Mikusinski's (1961) definition is that the associative law holds. Consequently, expressions such as $x^{-1}\cdot x\cdot\delta$ remain undefined, in view of (2.4).

This defect in Mikusinski's definition has been removed by Fisher (1971) by restricting Mikusinski's definition to binary operations. Thus, the product of three distributions $(f\cdot g)\cdot h$, even if it exists, is not necessarily equal to the limit of the sequence $f_n\cdot g_n\cdot h_n$, but, is given as the limit of the sequence $\rho_n\cdot h_n$, where ρ is the distribution $f\cdot g$. Fisher (1971) also obtains the formula (1.1) and other applications are to be found in Fisher (1972, 1973). The product, naturally, fails to be associative although it is commutative. The last two theories do not ascribe any general meaning to the symbol δ^2, and, hence, are not applicable to the sort of problems proposed in the introduction.

Thurber and Katz (1974) do not really define products using the non-standard extension, $^*D'$, of D'. Instead they seem to consider

$$\Delta^\rho(x-a) = (n/\pi)^{\rho/2}\exp[-n\rho(x-a)^2] \tag{2.7}$$

where n is a positive infinite constant, as a fractional power of the delta function. Naturally, there are various types of delta functions in this theory, i.e. the theory of Thurber and Katz deals with non-standard extensions of sequences converging to the delta distribution, and not with the delta distribution, *per se*.

2.2. *Definition of $f\cdot g$*

Consider the non-standard spaces $^*D$ and $^*D'$ (Stroyan and Luxemburg 1976). Define, for $f, g\in D'$

$$f_n = f\otimes\delta_n$$

$$\delta_n(x) = n\sigma(nx) \tag{2.8}$$

$$f\cdot g = \lim_{n=\omega}{}^*(f_n\cdot g) \tag{2.9}$$

σ being a symmetric, infinitely differentiable function with $\int_{-\infty}^{+\infty}\sigma(x)\,\mathrm{d}x = 1$, with $\sigma(0)\neq 0$, and with support contained in the interval $[-1, 1]$. The $*$ in (2.9) denotes the non-standard extension of the sequence of distributions $f_n\cdot g$, and the notation $\lim_{n=\omega}$ refers to an evaluation of the ωth term of this sequence for a fixed positive infinite integer ω. The non-standard representation of a given distribution is nearly unique, in that any two representations would differ by an infinitesimal distribution. If two (non-standard) distributions, h_1, h_2, differ by an infinitesimal distribution, we write $h_1 \doteq h_2$.

The product of two distributions, defined by (2.9), *always exists* in $^*D'$. In case f is a function, the product defined by (2.9) differs from the one defined by (2.1) by an infinitesimal distribution. (2.9) extends (2.1), and, in particular, we have

$$\delta^2 \doteq \delta_\omega(0)\delta. \tag{2.10}$$

δ^2 turns out to be an infinite distribution, i.e. $\langle\delta^2, g\rangle$ is infinite for $g\in \text{fin}\ {}^*D'$ (Stroyan and Luxemburg 1976).

Naturally, the choice of the infinite subscript ω and the sequence δ_n is non-unique, but different choices will lead to different distributions *only if $f \cdot g$ is infinite.* The last property is important because it ensures that the final result, which should be a standard one, would not be affected by the choice of δ_n and of ω. For problems relating to hypersurfaces of discontinuity, this comes about in the following manner: for standard real numbers a, b, c,

$$a\delta^2 + b\delta + c \triangleq 0 \tag{2.11}$$

iff

$$a = b = c = 0. \tag{2.12}$$

Proof. $c = 0$ trivially, and (2.11) implies that $a \triangleq -b/\delta_\omega(0)$ is an infinitesimal. Since a has been assumed to be a standard real number, $a = 0$, leading to $b = 0$.

When discontinuities are present in the field variables, equations of the type (1.2) and (1.3) would ultimately reduce to an equation of the form (2.11). Hence, the final result would be independent of the choice of δ_n and of ω.

We observe that this last property would not hold if we had defined $f \cdot g = f_\omega \cdot g_\omega$. Thus $\delta_\omega(x-a) \cdot \delta_\omega(x-b)$, $a \neq b$, would involve the product of an infinity and an infinitesimal, and would be infinite, finite or infinitesimal depending on the choice of the approximating sequence δ_n and the infinite integer ω. In contrast, with the definition (2.9), if $f \cdot g$ is infinite (finite, infinitesimal) for one choice of δ_n and ω, then it is infinite (finite, infinitesimal) for all choices of δ_n and ω.

In the present theory, also, it is possible to define fractional powers of the delta function by

$$\delta^\rho = \delta_\omega^{\rho-1}(0)\delta \tag{2.13}$$

leading, in particular, to the infinitesimal distribution

$$\delta^{1/2} = \delta_\omega^{-1/2}(0)\delta.$$

2.3. *Properties*

The commutative law fails, since

$$\delta\delta' \triangleq \lim_{n=\omega} \delta_n \cdot \delta' \triangleq -\delta'_\omega(0)\delta + \delta_\omega(0)\delta' \tag{2.14}$$

whereas

$$\delta'\delta \triangleq \delta'_\omega(0)\delta.$$

The associative law also fails, in general, since

$$(f\delta)\delta' \triangleq f(0)(\delta\delta') \triangleq -f(0)\delta'_\omega(0)\delta + f(0)\delta_\omega(0)\delta' \tag{2.15}$$

whereas

$$f(\delta\delta') \triangleq -[f(0)\delta'_\omega(0) + f'(0)]\delta + f(0)\delta_\omega(0)\delta'. \tag{2.16}$$

The failure of the commutative and associative laws is unimportant within the present-day symmetric formalism of quantum field theory (Guttinger 1955). Therefore, for the purposes of quantum field theory, the product may even be symmetrised by defining

$$f \odot g = \tfrac{1}{2}(f \cdot g + g \cdot f). \tag{2.17}$$

In view of (2.4), it is not possible to obtain an associative product, since $f \cdot g$ always exists. Both the distributive laws are trivially valid.

As is obvious from (1.2) and (1.3), a situation frequently encountered in applications is the multiplication of a delta function by a discontinuous function. To cover this situation, we have the following theorem.

Theorem 1. If f is a function with a simple discontinuity at 0 then

$$f \cdot \delta \stackrel{*}{=} \tfrac{1}{2}[f(0^+) + f(0^-)]\delta \tag{2.18}$$

where $f \cdot \delta$ is defined by (2.9).

Proof. It is sufficient to prove that

$$(f \otimes \delta_\omega)(0) \stackrel{*}{=} \tfrac{1}{2}[f(0^+) + f(0^-)]. \tag{2.19}$$

Now, from (2.8)

$$(f \otimes \delta_n)(0) = \int_{-\infty}^{\infty} f(-y) n\sigma(ny)\,\mathrm{d}y \tag{2.20}$$

which gives, by a simple change of variables $(ny = x)$

$$(f \otimes \delta_n)(0) = \int_0^1 [f(-x/n) + f(x/n)]\sigma(x)\,\mathrm{d}x \tag{2.21}$$

since σ is symmetric, with support contained in $[-1, 1]$. Since f is continuous in a neighbourhood of zero

$$\begin{aligned} f(-x/n) &= f(0^-) + \varepsilon_1(n, x) \\ f(x/n) &= f(0^+) + \varepsilon_2(n, x) \end{aligned} \qquad 0 \leqslant x \leqslant 1 \tag{2.22}$$

where $|\varepsilon_1(n, x)| \leqslant \bar{\varepsilon}_1(n)$ and $|\bar{\varepsilon}_2(n, x)| \leqslant \bar{\varepsilon}_2(n)$, for $0 \leqslant x \leqslant 1$ and $\lim_{n\to\infty} |\bar{\varepsilon}_1(n)| = \lim_{n\to\infty} |\bar{\varepsilon}_2(n)| = 0$. Hence

$$|(f \otimes \delta_n)(0) - \tfrac{1}{2}[f(0^+) + f(0^-)]| \leqslant \bar{\varepsilon}_1(n) + \bar{\varepsilon}_2(n). \tag{2.23}$$

Since $\bar{\varepsilon}_1(\omega)$ and $\bar{\varepsilon}_2(\omega)$ are infinitesimals, the theorem holds.

Corollary 1. If H is the Heaviside function, $H(x) = 1$ for $x < 0$ and $H(x) = 0$ otherwise,

$$H \cdot \delta \stackrel{*}{=} \tfrac{1}{2}\delta. \tag{2.24}$$

(2.24) is also valid with Fisher's (1971) definition.

Theorem 2 (Leibniz rule). For $f, g \in D'$,

$$(f \cdot g)^{(k)} \stackrel{*}{=} \sum_{i=0}^{k} \binom{k}{i} f^{(i)} g^{(k-i)}. \tag{2.25}$$

Proof. By definition, $(f \cdot g)^{(k)} \stackrel{*}{=} (f_\omega \cdot g)^{(k)}$. Since f_ω is a function, the validity of the Leibniz rule for the product (2.1) implies that

$$(f_\omega \cdot g)^{(k)} \stackrel{*}{=} \sum_{i=0}^{k} \binom{k}{i} f_\omega^{(i)} \cdot g^{(k-i)}. \tag{2.26}$$

Since, by definition,

$$f^{(i)} \cdot g^{(k-i)} = f_\omega^{(i)} \cdot g^{(k-i)},$$

(2.25) holds.

Corollary 2. If H is the Heaviside function,

$$H \cdot \delta' \stackrel{*}{=} \tfrac{1}{2}\delta' - \delta^2. \tag{2.27}$$

Proof. $(H \cdot \delta)' \stackrel{*}{=} (\tfrac{1}{2}\delta)' \stackrel{*}{=} H' \cdot \delta + H \cdot \delta' \stackrel{*}{=} \delta^2 + H \cdot \delta'$.

Theorem 3. If a_{ij}, b_k, and c are standard real numbers

$$\sum a_{ij}\delta^{(i)}(x-x_0)\delta^{(j)}(x-x_0) + \sum b_k\delta^{(k)}(x-x_0) + c \stackrel{*}{=} 0 \tag{2.28}$$

iff $a_{ij} = b_k = c = 0$.

Proof. The proof is similar to that of (2.11)–(2.12) and will be omitted.

2.4. $\delta^{(k)}x^{-n}$

Products of the form $\delta^{(k)} \cdot x^{-n}$ are useful in quantum renormalisation theory, and these products are evaluated below. To begin with, it is necessary to distinguish the functions x^{-n} from the corresponding distributions, which are defined as follows (Gel'fand and Shilov 1964): for $g \in D$

$$\langle x^{-1}, g\rangle = PV\int \frac{g(x)}{x}\,\mathrm{d}x = \lim_{\varepsilon\to 0}\int_{x>|\varepsilon|} \frac{g(x)}{x}\,\mathrm{d}x \tag{2.29}$$

$$m!\,x^{-m-1} = (-1)^m \frac{\mathrm{d}^m}{\mathrm{d}x^m} x^{-1} \qquad m \geq 1. \tag{2.30}$$

Now, $\langle \delta^{(k)} \cdot x^{-n}, g\rangle$ is finite provided $\lim_{x\to 0}[g(x) \cdot x^{-n}]^{(k)}$ exists and, in that case,

$$\begin{aligned}\langle \delta^{(k)} \cdot x^{-n}, g\rangle &\stackrel{*}{=} \lim_{x\to 0}(-1)^k[g(x)\cdot x^{-n}]^{(k)} \\ &\stackrel{*}{=} \lim_{x\to 0}(-1)^k \sum_{i=0}^{k}\binom{k}{i}(-1)^i\frac{(n+i-1)!}{(n-1)!}\frac{g^{(k-i)}}{x^{n+i}}.\end{aligned} \tag{2.31}$$

The limit of any of the terms in the above summation exists, iff $g \in {}^*D(0, 1, \ldots, k+n-1) = {}^*\{\varphi \in D, \varphi(0) = \varphi'(0) = \ldots = \varphi^{(k+n-1)}(0) = 0\}$, and in that case

$$\lim_{x\to 0}\frac{g^{(k-i)}(x)}{x^{n+i}} \stackrel{*}{=} \frac{g^{(k+n)}(0)}{(n+i)!}. \tag{2.32}$$

That is

$$\langle \delta^{(k)} \cdot x^{-n}, g\rangle \stackrel{*}{=} \frac{(-1)^k g^{(k+n)}(0)}{(n-1)!}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^i}{n+i} \qquad g \in {}^*D(0, 1, \ldots, k+n-1). \tag{2.33}$$

It follows that

$$\delta^{(k)} \cdot x^{-n} \stackrel{*}{=} \sum_{i=1}^{k+n-1} c_{ki}^{n} \delta^{(i)} + \frac{(-1)^n k!}{(k+n)!} \delta^{(k+n)}. \tag{2.34}$$

To evaluate the constants $c_{ki}^n \in {}^*\mathbb{R}$, we select functions $h_i(x)$, that behave like $x^i/i!$ in a neighbourhood of zero, so that

$$\begin{aligned} &h_i^{(k)}(0) = 0 \qquad k \neq i \\ &h_i^{(i)}(0) = 1 \end{aligned} \tag{2.35}$$

and

$$\langle \delta^{(k)} \cdot x^{-n}, h_i \rangle \stackrel{*}{=} (-1)^i c_{ki}^n. \tag{2.36}$$

But, by definition,

$$\langle \delta^{(k)} \cdot x^{-n}, h_i \rangle = \lim_{j=\omega} \langle \delta_j^{(k)} \cdot x^{-n}, h_i \rangle. \tag{2.37}$$

For large j

$$\begin{aligned} \langle \delta_j^{(k)} \cdot x^{-n}, h_i \rangle &= \langle \delta_j^{(k)} \cdot x^{-n}, x^i/i! \rangle \\ &= (1/i!) \langle x^{-n+i}, \delta_j^{(k)} \rangle \qquad (2.38) \\ &= 0 \qquad \text{if } n \leqslant i. \qquad (2.39) \end{aligned}$$

If $i \leqslant n-1$

$$\lim_{j=\omega} \frac{1}{i!} \langle x^{-n+i}, \delta_j^{(k)} \rangle \stackrel{*}{=} \lim_{j=\omega} \frac{(-1)^k}{i!} \left\langle \frac{\mathrm{d}^k}{\mathrm{d}x^k} x^{-n+i}, \delta_j \right\rangle \stackrel{*}{=} \frac{(n+k-i-1)!}{i!(n-i-1)!} x_\omega^{-n-k+i}(0) \tag{2.40}$$

where

$$x_\omega^{-n}(0) \stackrel{*}{=} \lim_{j=\omega} (x^{-n} \otimes \delta_j)(0) \stackrel{*}{=} \lim_{j=\omega} \langle x^{-n}, \delta_j \rangle. \tag{2.41}$$

Consequently,

$$\delta^{(k)} \cdot x^{-n} \stackrel{*}{=} \sum_{i=0}^{n-1} (-1)^i \frac{(n+k-i-1)!}{i!(n-i-1)!} x_\omega^{-n-k+i}(0) \delta^{(i)} + \frac{(-1)^n k! \delta^{(n+k)}}{(n+k)!} \tag{2.42}$$

where $x_\omega^{-n-k+i}(0) = 0$ if $n+k-i$ is odd.

The products $x^{-n} \cdot \delta^{(k)}$ are different, and can be evaluated using the identity

$$f \cdot \delta^{(k)} \stackrel{*}{=} \sum_{i=0}^{k} (-1)^i \binom{k}{i} f^{(k-i)}(0) \delta^{(i)} \tag{2.43}$$

for an infinitely differentiable function f. Thus,

$$\begin{aligned} x^{-n} \cdot \delta^{(k)} &\stackrel{*}{=} \lim_{j=\omega} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (x_j^{-n})^{(k-i)}(0) \delta^{(i)} \\ &\stackrel{*}{=} (-1)^k \sum_{i=0}^{k} \binom{k}{i} \frac{(n+k-1-i)!}{(n-1)!} x_\omega^{-n-k+i}(0) \delta^{(i)} \qquad (2.44) \end{aligned}$$

where, as before, $x_\omega^{-n-k+i}(0) = 0$ if $n+k-i$ is odd.

For the case $k=0$, $n=1$, we obtain

$$\delta \cdot x^{-1} \stackrel{*}{=} -\delta' \qquad x^{-1} \cdot \delta \stackrel{*}{=} 0 \tag{2.45}$$

so that

$$\delta \odot x^{-1} \stackrel{*}{=} -\tfrac{1}{2}\delta'. \tag{2.46}$$

Products with other extensions of x^{-n} may also be obtained from the above formulae. Thus,

$$\delta \cdot (x \pm \mathrm{i}\varepsilon)^{-1} \stackrel{*}{=} -\delta' \mp \pi \mathrm{i} \delta^2 \tag{2.47}$$

where

$$(x \pm \mathrm{i}\varepsilon)^{-1} \stackrel{*}{=} x^{-1} \mp \pi \mathrm{i} \delta. \tag{2.48}$$

3. Compositions

If g is a C^1 function, $g(x_1)=0$, $g'(x_1) \neq 0$, $\delta(g(x))$ is usually defined (Gel'fand and Shilov 1964) by carrying out a formal change of variables

$$\langle \delta(g(x)), h \rangle = h(x_1)/|g'(x_1)| \qquad \forall h \in D$$

i.e.

$$\delta(g(x)) = |g'(x_1)|^{-1}\delta(x-x_1). \tag{3.1}$$

Here we shall define, for any $f \in D'$, $g \in C^\infty$, $g' \neq 0$

$$f(g(x)) = \lim_{n=\omega} {}^{*} (f_n(g(x))) \qquad f_n = f \otimes \delta_n. \tag{3.2}$$

The distribution defined by (3.2) always exists and is given, using the change of variables formula for ordinary functions, by

$$\langle f(g(x)), h \rangle \stackrel{*}{=} \left\langle f_\omega, \frac{h(g^{-1}(x))}{|g'(g^{-1}(x))|} \right\rangle \stackrel{*}{=} \left\langle f, \frac{h(g^{-1}(x))}{|g'(g^{-1}(x))|} \right\rangle. \tag{3.3}$$

Moreover, the distribution defined by (3.2) is nearly unique in that a different sequence δ_n or a different infinite constant ω would lead to a distribution $f_{\omega'}(g(x))$ which would differ from $f_\omega(g(x))$ by an infinitesimal distribution, provided $f(g(x))$ is finite.

Theorem 4 (*chain rule*). If $f \in D'$, $g \in C^\infty$, $g' \neq 0$,

$$[f(g(x))]' \stackrel{*}{=} f'(g(x)) \cdot g'(x). \tag{3.4}$$

Proof. The non-standard proof follows immediately from the usual chain rule. One can also see this directly, since for any $h \in D$,

$$\begin{aligned} \langle [f(g(x))]'\, h \rangle &\stackrel{*}{=} -\langle f(g(x)), h' \rangle \\ &\stackrel{*}{=} -\left\langle f, \frac{h'(g^{-1}(x))}{|g'(g^{-1}(x))|} \right\rangle \end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\langle f'(g(x))\cdot g'(x), h\rangle &\overset{*}{=} \langle f'(g(x)), g'(x)\cdot h\rangle \\ &\overset{*}{=} \left\langle f', \frac{g'(g^{-1}(x))\cdot h(g^{-1}(x))}{|g'(g^{-1}(x))|}\right\rangle \\ &\overset{*}{=} \langle -f, \pm[h(g^{-1}(x))]'\rangle \\ &\overset{*}{=} -\left\langle f, \frac{h'(g^{-1}(x))}{|g'(g^{-1}(x))|}\right\rangle \end{aligned} \tag{3.6}$$

since $(g^{-1})' = 1/g'(g^{-1})$.

In case g has several real roots $x_1, x_2, \ldots, x_n$, $g'(x_i) \neq 0$, the definition (3.2) reduces to the usual definition for a distribution of finite order, provided $g'(g^{-1}(x)) \neq 0$, for $x \in \operatorname{supp} f$, since

$$\langle \delta_\omega^{(k)}(g(x)), h\rangle \overset{*}{=} \sum_{i=1}^{n} (-1)^k \left[\frac{h(g^{-1}(x))}{|g'(g^{-1}(x))|}\right]^{(k)}_{x=x_i}. \tag{3.7}$$

The chain rule continues to be valid.

3.1. Multiple roots

In case $g \in C^\infty$ has multiple roots at x_1, then (3.3) can no longer be used, since $g'(x_1) = 0$. In this situation, Fisher (1974) and Tewari (1977) have adopted a limiting procedure to define $\delta^{(r)}(x^{2m+1})$, yielding

$$\delta^{(r)}(x^{2m+1}) = \frac{r!\,\delta^{2mr+2m+r}(x)}{(2m+1)(2mr+2m+r)!} + \sum_{i=0}^{2mr+2m+r-1} C_i^{rm}\delta^{(i)} \tag{3.8}$$

where C_i^{rm} are arbitrary constants, the functional having been extended in the usual manner. The distributions corresponding to $\delta^{(r)}(x^{2m})$ are not defined by Fisher (1974), because x^{2m} is not invertible in a neighbourhood of zero.

If the definition (3.2) is used, $\delta^{(r)}(g(x))$ is defined for $g \in C^\infty$, regardless of the nature of the roots of $g(x)$. Thus,

$$\langle \delta^{(r)}(g(x)), \varphi(x)\rangle \overset{*}{=} \lim_{n=\omega} \int_{-\infty}^{\infty} \delta_n^{(r)}(g(x))\varphi(x)\,\mathrm{d}x \tag{3.9}$$

and if

$$I_n^r = \int_{-\infty}^{\infty} \delta_n^{(r)}(g(x))\varphi(x)\,\mathrm{d}x \tag{3.10}$$

then I_ω^r is $*$ finite, since I_n^r is finite, the integrand being continuous with compact support. If g is invertible in a neighbourhood of zero, I_ω^r is finite provided $\lim_{x\to 0} \varphi(g^{-1}(x))/g'(g^{-1}(x))$ exists. In particular, if $g(x) = x^{2m+1}$, the definition (3.2) agrees with (3.8) on the subspace $D(0, 1, \ldots, 2mr+2m+r-1)$. However, the constants C_i^{rm}, in this theory, are not arbitrary. To evaluate these constants, we select, as in (2.35), functions $h_i \in D$ which behave like $x^i/i!$ in a neighbourhood of zero. Thus,

$$\langle \delta^{(r)}(x^{2m+1}), h_i(x)\rangle \overset{*}{=} (-1)^i C_i^{rm} \overset{*}{=} \lim_{n=\omega} \int_{-\infty}^{\infty} \delta_n^{(r)}(x^{2m+1})h_i(x)\,\mathrm{d}x. \tag{3.11}$$

For large n

$$I_n = \int_{-\infty}^{\infty} \delta_n^{(r)}(x^{2m+1}) h_i(x)\, \mathrm{d}x$$

$$= \int_{-\infty}^{\infty} \delta_n^{(r)}(x^{2m+1}) \frac{x^i}{i!}\, \mathrm{d}x \tag{3.12}$$

$$= \int_{-\infty}^{0} + \int_{0}^{\infty}. \tag{3.13}$$

Putting $y = -x$ in the first integral in (3.13), we obtain

$$I_n = \frac{1}{i!} \int_0^{\infty} \delta_n^{(r)}(x^{2m+1})[x^i + (-1)^{(r+i)} x^i]\, \mathrm{d}x. \tag{3.14}$$

So $I_n = 0$ if $i + r$ is odd, and otherwise

$$I_n = \frac{2}{i!} \int_0^{\infty} \delta_n^{(r)}(x^{2m+1}) x^i\, \mathrm{d}x. \tag{3.15}$$

Substituting $z = x^{2m+1}$

$$I_n = \frac{2}{i!(2m+1)} \int_0^{\infty} \delta_n^{(r)}(z) z^{(i-2m)/(2m+1)}\, \mathrm{d}z \tag{3.16}$$

$$= \frac{2}{i!(2m+1)} \int_{-\infty}^{\infty} \delta_n^{(r)}(z) z_+^{(i-2m)/(2m+1)}\, \mathrm{d}z$$

$$= \frac{(-1)^r 2}{i!(2m+1)^{r+1}} \prod_{j=0}^{r} (2mj + 2m + j - i)(z_+^{(i-2mr\varepsilon 2m-r)/(2m+1)} \otimes \delta_n)(0) \tag{3.17}$$

where $z_+^{\beta} = z^{\beta} H(z)$, H being the Heaviside function. Hence,

$$i!\, C_i^{rm} \doteq 2(2m+1)^{-r-1} \prod_{j=1}^{r} (2mj + 2m + j - i)(z_{+\omega}^{(i-2mr-2m-r)/(2m+1)})(0). \tag{3.18}$$

One may also evaluate the C_i^{rm} explicitly, in terms of σ and ω, by observing that the limits of integration in (3.12) are from $-n^{-1/(2m+1)}$ to $n^{-1/(2m+1)}$ and substituting $y = n^{1/(2m+1)} x$ to obtain

$$i!\, C_i^{rm} \doteq (-1)^i 2 \omega^{(2mr+2m+r-i)/(2m+1)} \int_{-1}^{1} \sigma^{(r)}(y^{2m+1}) y^i\, \mathrm{d}y$$

$$1 \leqslant i \leqslant 2mr + 2m + r - 1,\ i + r \text{ even}. \tag{3.19}$$

3.2. Even functions

Compositions with even functions occur in many situations in physics. Apart from situations arising out of the examples mentioned in the introduction, we may mention the Schwarzschild–Tetrode–Fokker action (Hoyle and Narlikar 1964)

$$J = -\sum_i m_i \int (\dot{z}_{(i)}^{\mu} \dot{z}_{(i)\mu})^{1/2}\, \mathrm{d}\tau_i - \tfrac{1}{2} \sum_{i \neq j} e_i e_j \int \dot{z}_{(i)}^{\nu} \delta[(z_{(i)}^{\mu} - z_{(j)}^{\mu})(z_{(i)\mu} - z_{(j)\mu})] \dot{z}_{\nu}^{(j)}\, \mathrm{d}\tau_i\, \mathrm{d}\tau_j \tag{3.20}$$

where e_i is the charge, m_i the mass and $z_i = z_i(\tau_i)$ the world line of the ith particle.

As pointed out earlier, compositions with even functions have not been defined previously because a change of variables cannot immediately be carried out. However, expressions such as $\delta_\omega^{(r)}(x^{2m})$ still make sense, and these still induce distributions in ${}^*D'$ because $\delta_n^{(r)}(x^{2m})$ induces a distribution for each n. For example,

$$\langle\delta_\omega(x^{2m}), g\rangle \triangleq \lim_{n=\omega}\int_{-\infty}^{\infty}\delta_n(x^{2m})g(x)\,dx$$

$$\triangleq \lim_{n=\omega}\int_{-\infty}^{\infty} n\sigma(nx^{2m})g(x)\,dx$$

$$\triangleq \lim_{n=\omega}\int_{-n^{-1/2m}}^{n^{-1/2m}} n\sigma(nx^{2m})g(x)\,dx \tag{3.21}$$

since $\operatorname{Supp}\sigma\subseteq[-1,1]$. Now

$$I_n=\int_{-n^{-1/2m}}^{n^{-1/2m}} n\sigma(nx^{2m})g(x)\,dx=\int_{-n^{-1/2m}}^{0}+\int_{0}^{n^{-1/2m}}. \tag{3.22}$$

In the first integral we carry out the change of variables $y=-x$, yielding

$$I_n=\int_0^{n^{-1/2m}} n\sigma(nx^{2m})[g(x)+g(-x)]\,dx. \tag{3.23}$$

Substituting $x=(z/n)^{1/2m}$, $0\leqslant x\leqslant n^{-1/2m}$

$$I_n=\frac{1}{2m}\int_0^1\sigma(z)\{g[(z/n)^{1/2m}]+g[-(z/n)^{1/2m}]\}(z/n)^{(1-2m)/2m}\,dz. \tag{3.24}$$

The integral in (3.20) is finite provided

$$\lim_{y\to0^+}[g(y^{1/2m})+g(-y^{1/2m})]\cdot y^{(1-2m)/2m}$$

exists. The limit in question exists and is zero provided $g\in D(0,1,2,\ldots,2m-2)$. Since g has compact support, $\lim_{n\to\infty}I_n=0$, i.e. $\delta_\omega(x^{2m})$ is infinitesimal for $g\in$ fin ${}^*D(0,1,2,\ldots,2m-2)$. It follows that

$$\delta_\omega(x^{2m})\triangleq\sum_{i=0}^{2m-2}a_i^{0m}\delta^{(i)}. \tag{3.25}$$

More generally, the above procedure yields

$$\delta^{(r)}(x^{2m})\triangleq\sum_{i=0}^{2mr+2m-2}a_i^{rm}\delta^{(i)} \tag{3.26}$$

where the constants a_i^{rm} may be evaluated as before

$$a_i^{rm}\triangleq 0 \qquad \text{if } i+r \text{ even} \tag{3.27}$$

$$i!\,a_i^{rm}\triangleq -2(2m)^{-r-1}\prod_{j=0}^{r}(2mj+2m-1-i)(z_{+\omega}^{(i-2mr-2m+1)/2m})(0) \tag{3.28}$$

$$\triangleq(-1)^i 2\omega^{(2mr+2m-i-1)/2m}\int_0^1\sigma^{(r)}(z^{2m})z^i\,dz$$
$$1\leqslant i\leqslant 2mr+2m-2,\ i+r \text{ odd}. \tag{3.29}$$

In view of (3.29), all the distributions $\delta^{(r)}(x^{2m})$ are infinite.

Compositions with oscillating functions also occur, for instance, in the study of oscillating surface layers in relativity. For this purpose, we record

$$\delta(1-\sin x) \triangleq k \sum_n \delta(x - \tfrac{1}{2}(4n+1)\pi) \tag{3.30}$$

$$k \triangleq \frac{\omega}{2} \int_0^{\frac{1}{2}\pi} \sigma(1-\sin y)\sqrt{1+\sin y}\, \mathrm{d}y. \tag{3.31}$$

Before concluding this section, we observe that the hypothesis $g \in C^\infty$ is not essential for defining $f(g(x))$. For $f \in D'$, $f(g(x))$ exists in $^*D'$ provided $f_n(g(x)) \in D'$ for each n, i.e. provided $f_n(g(x))$ is locally integrable.

4. Applications

4.1. Derivation of the Rankine–Hugoniot equations

It is usually asserted that the Euler equations (1.3) are not valid at discontinuities, where one must use, for instance, the Rankine–Hugoniot equations. The Rankine–Hugoniot equations are usually stated in the form

$$\rho_0(U-u_0) = \rho(U-u) \qquad \rho_0(U-u_0)(u-u_0) = P - P_0 \tag{4.1}$$

where U is the shock velocity, and the subscript 0 refers to the undisturbed fluid ahead of the shock. We propose to derive these equations from the usual equations of continuity and momentum (1.3) thereby demonstrating that the equations (1.3) are indeed valid at discontinuities. To this end, we observe that the equations (4.1) are valid for normal shocks of infinite extent, and a two-point flow field in two dimensions, with a simple discontinuity at the surface of the shock.

To derive these equations, using the present methods, we suppose that the hypersurface of discontinuity is given by

$$y = \bar{y}(t) = Ut \qquad U = \partial\bar{y}/\partial t = \text{constant}. \tag{4.2}$$

Further, let

$$\begin{aligned} \rho &= \rho^-\chi_- + \rho^+\chi_+ \qquad u_i = u_i^-\chi_- + u_i^+\chi_+ \qquad i = 1, 2 \\ P &= P^-\chi_- + P^+\chi_+ \end{aligned} \tag{4.3}$$

where

$$\chi_+ = \chi_{(\bar{y}(t),\infty)}(x) = H(x - \bar{y}(t)) \qquad \chi_- = 1 - \chi_+ \tag{4.4}$$

H being the Heaviside function.

We observe that

$$\begin{aligned} \frac{\partial\chi_+}{\partial x} &= \delta(x-\bar{y}(t)) \qquad \frac{\partial\chi_-}{\partial x} = -\delta(x-\bar{y}(t)) \\ \frac{\partial\chi_+}{\partial t} &= -U\delta(x-\bar{y}(t)) \qquad \frac{\partial\chi_-}{\partial t} = +U\delta(x-\bar{y}(t)). \end{aligned} \tag{4.5}$$

The equation of continuity, in two dimensions, is

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u_1)}{\partial x} + \frac{\partial(\rho u_2)}{\partial y} = 0. \tag{4.6}$$

Substituting (4.3) in (4.6), we have

$$\rho^- \frac{\partial \chi_-}{\partial t} + \rho^+ \frac{\partial \chi_+}{\partial t} + \rho^- u_1^- \frac{\partial \chi_-}{\partial x} + \rho^+ u_1^+ \frac{\partial \chi_+}{\partial x} = 0. \tag{4.7}$$

Substituting (4.5) in (4.7) leads to

$$-(\rho^- - \rho^+)U\delta(x - \bar{y}(t)) = (p^+ u_1^+ - p^- u_1^-)\delta(x - \bar{y}(t)). \tag{4.8}$$

Hence, by proposition 3,

$$\rho^+(U - u_1^+) = \rho^-(U - u_1^-) \tag{4.9}$$

which is the same as the first of (4.1) with $\rho_0 = \rho^+$, $\rho = \rho^-$, $u_0 = u_1^+$, $u = u_1^-$.

The momentum equation is

$$\rho \frac{\partial u_1}{\partial t} + \rho u_1 \frac{\partial u_1}{\partial x} = -\frac{\partial P}{\partial x} - \frac{\partial P}{\partial y}. \tag{4.10}$$

Substituting (4.3) and (4.5) in (4.10), we have

$$\begin{aligned} -(Uu_1^+ - Uu_1^-)\rho\delta(x - \bar{y}(t)) + (\rho^+ u_1^+ \chi_+ + \rho^- u_1^- \chi_-)(u_1^+ - u_1^-)\delta(x - \bar{y}(t)) \\ = (P^- - P^+)\delta(x - \bar{y}(t)). \end{aligned} \tag{4.11}$$

Hence, by theorems 1 and 3,

$$\tfrac{1}{2}(u_1^+ - u_1^-)[\rho^+ u_1^+ + \rho^- u_1^- - \rho^+ U - \rho^- U] = P^- - P^+. \tag{4.12}$$

Using (4.9), (4.12) leads to the second Rankine–Hugoniot equation

$$\rho^+(U - u_1^+)(u_1^- - u_1^+) = P^- - P^+. \tag{4.13}$$

Sometimes a third Rankine–Hugoniot equation is used, and this can be derived similarly from the energy equation.

From the physical point of view, the equations (1.3) only express the conservation of mass and momentum, in differential form, and it is possible to derive the Rankine–Hugoniot equations by appealing directly to these conservation principles. Alternatively, one may adopt the procedure of expressing the physical laws underlying (1.3) in integral form in a narrow region enclosing the shock, and use the divergence theorem to pass to the limit. This corresponds to our earlier assertion that any standard result derived using non-standard techniques could, in principle, have been derived without using these techniques. However, the merits of the non-standard procedure become obvious when we try to integrate the equations (1.3) for the case of an arbitrary hypersurface, or in relativity, for the case where the $g_{\mu\nu}$ are discontinuous.

4.1.1. The effect of viscosity. If turbulence is present behind, and in front of, the shock, the effect of (eddy) viscosity may be taken into account by including a term of the form $\mu \nabla^2 u$ in place of the viscous stress tensor in (1.3). Now, (4.5) yields

$$\partial u_1/\partial x = (u_1^+ - u_1^-)\delta(x - \bar{y}(t)) \qquad \partial^2 u_1/\partial x^2 = (u_1^+ - u_1^-)\delta'(x - \bar{y}(t)). \tag{4.14}$$

Hence, by theorem 3, we would have an additional equation of the form

$$\mu(u_1^+ - u_1^-) = 0 \tag{4.15}$$

which must be regarded as a consistency condition. Using equation (4.13) this can be put in the form

$$N=\frac{\mu(P^- - P^+)}{\rho^+(u_1^+ - U)}=0. \tag{4.16}$$

If we only have $N \simeq 0$, the Rankine–Hugoniot equations are only approximately applicable.

The physical interpretation of this consistency condition is that the Rankine–Hugoniot equations are not valid for strong shocks—strong shocks would curve due to the effect of viscosity. It is proposed to develop a general theory of curved shocks, using the above methods.

Applications to the problem of junction conditions have been considered separately (Raju 1982).

4.2. Calculation of transition probabilities

In quantum field theory, to calculate the probabilities of various scattering processes, it is necessary to evaluate squares of the *S*-matrix elements. The *S*-matrix elements may be written in the general form (Bogoliubov and Shirkov 1959, equation (21.35))

$$\Phi_{\dots p'\dots} S\Phi_{\dots p\dots} = \delta(\textstyle\sum p - \sum p')F(p', p) \tag{4.17}$$

where $\Phi_{\dots p\dots}$ denotes the initial, s-particle state $\Phi_{p_1\sigma_1 p_2\sigma_2\dots p_s\sigma_s}$ and $\Phi_{\dots p'\dots}$ denotes the final state $\Phi_{p'_1\sigma_1 p'_2\sigma_2\dots p'_r\sigma_r}$ the *i*th particle having momentum p_i and inner quantum number σ_i. The appearance of the (4-dimensional) δ function signifies the conservation of energy and momentum. If the states are normalised to one particle per unit volume, it follows that (Bogoliubov and Shirkov 1959, equation (22.14)) the numbers of particles scattered into the momentum intervals $\Delta \boldsymbol{p}_1, \Delta \boldsymbol{p}_2, \dots, \Delta \boldsymbol{p}_r$ are given by

$$(2\pi)^{3s}|\Phi_{\dots p'\dots}S\Phi_{\dots p\dots}|^2 \Delta \boldsymbol{p}_1 \Delta \boldsymbol{p}_2 \dots \Delta \boldsymbol{p}_r. \tag{4.18}$$

For the purpose of this discussion, we may and do assume that $F(p', p)$ is continuous. Nevertheless, the expression (4.18) turns out to be infinite because of the appearance of the square of the δ function. To obtain a correspondence with experiments, it is clearly necessary that the differential cross section calculated using (4.18) should be finite. Consequently, it is customary (for instance, Schweber 1964, Bjorken and Drell 1965) to follow the procedure proposed by Lippmann and Schwinger (1950) and divide by the apparently meaningless quantity $(2\pi)^4\delta(0)$.

We propose, heuristically, the following justification for the procedure of dividing out by the infinite, but meaningful, quantity $(2\pi)^4\delta_\omega(0)$. The squares of the *S*-matrix elements cannot be regarded as probabilities, because a probability must lie between 0 and 1 and should *sum up to 1.* The squares of the *S*-matrix elements, however, can be regarded as *proportional* to the probability of transition from the initial to the final state. To obtain the relevant probability, it is only necessary to divide by the constant of proportionality—in this case, the infinite number $(2\pi)^4\delta_\omega(0)$. To interpret the probability so obtained, one may go over to the usual procedure of switching on the interaction adiabatically with an intensity *g*, in a region of volume *V* and for time *T*. In the limiting case as $g \to 1$, division by $(2\pi)^4\delta\omega(0)$ corresponds to division by *VT*, so that the transition probability represents the fraction of incident particles scattered per unit volume per unit time.

Thus, while the justification for division by $(2\pi)^4\delta_\omega(0)$ comes from the requirement that a probability should sum up to give 1, the interpretation of the resulting probability may be represented symbolically, as it usually is, by $(2\pi)^4\delta(0)=VT$.

The problem of renormalisation arises because of singularities of the function $F(p',p)$, corresponding to the infinities in the products of the Green functions, and perhaps a similar procedure of division would yield the desired results in this case as well.

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